

# Special features of the relation between Fisher Information and Schrödinger eigenvalue equation

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## Abstract

It is well known that a suggestive relation exists that links Schrödinger's equation (SE) to the information-optimizing principle based on Fisher's information measure (FIM). The connection entails the existence of a Legendre transform structure underlying the SE. Here we show that appeal to this structure leads to a first order differential equation for the SE's eigenvalues that, in certain cases, can be used to obtain the eigenvalues without explicitly solving SE. Complying with the above mentioned equation constitutes a necessary condition to be satisfied by an energy eigenvalue. We show that the general solution is unique.

KEYWORDS: Information Theory, Fisher's Information measure, Legendre transform and Virial theorem.

## 1. INTRODUCTION

It is well-known that a strong link exists between Fisher's information measure (FIM)  $I$  and Schrödinger wave equation (SWE) [1–5]. In a nutshell, this connection is based upon the fact that the constrained minimization of  $I$  leads to a SWE [1–5]. This, in turn, implies intriguing relationships between various aspects of SWE, on the one hand, and the formalism of statistical mechanics as derived from Jaynes's maximum entropy principle, on the other one. In particular, fundamental consequences of the SWE, such as the Hellmann-Feynman and Virial theorems, can be re-interpreted in terms of a special kind of reciprocity relations between relevant physical quantities similar to the ones exhibited by the thermodynamics' formalism [4, 5]. This demonstrates that a Legendre-transform structure underlies the non-relativistic Schrödinger equation. In this communication we show that such structure allows one to obtain a first-order differential equation that energy eigenvalues must necessarily satisfy.

## 2. BASIC IDEAS

Fisher Information measure has been successfully applied to the study of several physical scenarios, particularly quantum mechanical ones (as a non-exhaustive recent set, see for instance [7–15]). We will briefly review here the pertinent formalism. If an observer were to make a measurement of  $x$  and had to best infer  $\theta$  from such measurement, calling the resulting estimate  $\tilde{\theta} = \tilde{\theta}(x)$ , one might wonder how well  $\theta$  could be determined. Estimation theory [6] asserts that the *best possible estimator*  $\tilde{\theta}(x)$ , after a very large number of  $x$ -samples is examined, suffers a mean-square error  $e^2$  from  $\theta$  obeying the rule  $Ie^2 = 1$ , where the Fisher information measure (FIM)  $I$ , a functional of the PDF, reads

$$I = \int dx f(x, \theta) \left\{ \frac{\partial}{\partial \theta} \ln [f(x, \theta)] \right\}^2. \quad (1)$$

Any other estimator must have a larger mean-square error (all estimators must be unbiased, i.e., satisfy  $\langle \tilde{\theta}(\mathbf{x}) \rangle = \theta$ ). Thus, FIM has a lower bound. No matter what the parameter  $\theta$  of the system might be,  $I$  has to obey

$$I e_{\theta}^2 \geq 1, \quad (2)$$

the celebrated Cramer–Rao bound [6]. The particular instance of translational families merits special consideration. These are mono-parametric distribution families of the form  $f(x, \theta) = f(x - \theta)$ , known up to the shift parameter  $\theta$ . All family members exhibit identical shape. After introducing the amplitudes  $\psi$  such that the probability distribution function (PDF) are expressed via  $f(x) = \psi(x)^2$ , FIM adopts the simpler aspect [8]

$$I = \int dx f(x) \left\{ \frac{\partial}{\partial x} \ln [f(x)] \right\}^2 = 4 \int dx [\psi'(x)]^2; \quad (d\psi/dx = \psi'). \quad (3)$$

Note that for the uniform distribution  $f(x) = \text{constant}$  one has  $I = 0$ . Focus attention now a system that is specified by a set of  $M$  physical parameters  $\mu_k$ . We can write  $\mu_k = \langle A_k \rangle$  with  $A_k = A_k(x)$ . The set of  $\mu_k$ -values is to be regarded as our prior knowledge. It represents available empirical information. Let the pertinent probability distribution function (PDF) be  $f(x)$ . Then,

$$\langle A_k \rangle = \int dx A_k(x) f(x), \quad k = 1, \dots, M. \quad (4)$$

In this context it can be shown (see for example [1, 3]) that the *physically relevant* PDF  $f(x)$  minimizes the FIM (3) subject to the prior conditions and the normalization condition.

In the celebrated MaxEnt approach of Jaynes [16] one *maximizes* the entropy, that behaves information-wise in opposite fashion to that of Fisher’s measure [8]. Normalization entails  $\int dx f(x) = 1$ , and, consequently, our Fisher-based extremization problem adopts the appearance

$$\delta \left( I - \alpha \int dx f(x) - \sum_{k=1}^M \lambda_k \int dx A_k(x) f(x) \right) = 0 \quad (5)$$

where we have introduced the  $(M + 1)$  Lagrange multipliers  $\lambda_k$  ( $\lambda_0 = \alpha$ ). In Ref. [1] one can find the details of how to go from (5) to a Schrödinger’s equation (SE) that yields the desired PDF in terms of the amplitude  $\psi(x)$  referred to above [i.e., before Eq. (3)]. This SE is of the form

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi - \sum_{k=1}^M \frac{\lambda_k}{8} A_k \psi = \frac{\alpha}{8} \psi, \quad (6)$$

and can be formally interpreted as the (real) Schrödinger equation for a particle of unit mass ( $\hbar = 1$ ) moving in the effective, “information-related pseudo-potential” [1]

$$U = U(x) = -\frac{1}{8} \sum_{k=1}^M \lambda_k A_k(x), \quad (7)$$

in which the normalization-Lagrange multiplier  $(\alpha/8)$  plays the role of an energy eigenvalue. The  $\lambda_k$  are fixed, of course, by recourse to the available prior information. Note that  $\psi(x)$  is always real in the case of one-dimensional scenarios, or for the ground state of a real potential in  $N$  dimensions [17]. In terms of the amplitudes  $\psi(x)$  we have

$$\begin{aligned} I &= \int dx f \left( \frac{\partial \ln f}{\partial x} \right)^2 = \int dx \psi_n^2 \left( \frac{\partial \ln \psi_n^2}{\partial x} \right)^2 = 4 \int dx \left( \frac{\partial \psi_n}{\partial x} \right)^2 = \\ &= -4 \int \psi_n \frac{\partial^2}{\partial x^2} \psi_n dx = \int \psi_n \left( \alpha + \sum_{k=1}^M \lambda_k A_k \right) \psi_n dx, \end{aligned}$$

i.e.,

$$I = \alpha + \sum_{k=1}^M \lambda_k \langle A_k \rangle. \quad (8)$$

a form that we will employ in our developments below. Some useful results of Refs. [4, 5] will be needed below. An essential ingredient in the present considerations is the *virial theorem* [18] that, of course, applies in this Schrödinger-scenario [19]. It states that

$$\left\langle -\frac{\partial^2}{\partial x^2} \right\rangle = \left\langle x \frac{\partial}{\partial x} U(x) \right\rangle. \quad (9)$$

The potential function  $U(x)$  belongs to  $\mathcal{L}_2$  and thus admit of a series expansion in  $x$ ,  $x^2$ ,  $x^3$ , etc. [19]. The  $A_k(x)$  themselves belong to  $\mathcal{L}_2$  as well and can be series-expanded in similar fashion. This enables us to base our future considerations on the assumption that the a priori knowledge refers to moments  $x^k$  of the independent variable, i.e.,

$$\langle A_k \rangle = \langle x^k \rangle, \quad (10)$$

and that one possesses information on  $M$  moment-mean values  $\langle x^k \rangle$ . Our “information” potential  $U$  then reads

$$U(x) = -\frac{1}{8} \sum_k \lambda_k x^k. \quad (11)$$

We will assume that the first  $M$  terms of the above series yield a satisfactory representation of  $U(x)$ . Consequently, the following identification is made

$$\text{Lagrange multipliers} \Leftrightarrow U(x)'s \text{ series - expansion's coefficients.} \quad (12)$$

Thus, Eq. (9) allows one to immediately obtain

$$\left\langle \frac{\partial^2}{\partial x^2} \right\rangle = \frac{1}{8} \sum_{k=1}^M k \lambda_k \langle A_k \rangle; \quad (A_k = x^k), \quad (13)$$

and thus, via (13) and the above mentioned relation  $I = -4 \left\langle \frac{\partial^2}{\partial x^2} \right\rangle$ , a useful, virial-related expression for Fisher's information measure can be arrived at

$$I = - \sum_{k=1}^M \frac{k}{2} \lambda_k \langle x^k \rangle, \quad (14)$$

which is an explicit function of the  $M$  physical parameters  $\langle x^k \rangle$  and their respective Lagrange multipliers (also,  $U(x)$ 's series-expansion's coefficients)  $\lambda_k$ . Eq. (14) encodes the information provided by the virial theorem [4, 5]. Note that if we define  $M$ -dimensional vectors  $\mathbf{X}$  of components  $X_k = \langle x^k \rangle$  and  $\mathbf{G}$  of components  $G_k = k \lambda_k / 2$  we can cast  $I$  in the scalar-product fashion

$$I = -\mathbf{X} \cdot \mathbf{G}. \quad (15)$$

### 3. THE LEGENDRE STRUCTURE

The connection between our variational solutions  $f$  and thermodynamics was established in Refs. [1] and [2] in the guise of reciprocity relations that express the Legendre-transform structure of thermodynamics. They constitute its essential formal ingredient [20] and were re-derived à la Fisher in [1] by recasting (8) in a fashion that emphasizes the role of the relevant independent variables

$$I(\langle A_1 \rangle, \dots, \langle A_M \rangle) = \alpha + \sum_{k=1}^M \lambda_k \langle A_k \rangle. \quad (16)$$

The Legendre transform changes the identity of our relevant variables. As for  $I$  we have

$$\alpha = I(\langle A_1 \rangle, \dots, \langle A_M \rangle) - \sum_{k=1}^M \lambda_k \langle A_k \rangle = \alpha(\lambda_1, \dots, \lambda_M), \quad (17)$$

so that we encounter the three reciprocity relations proved in [1]

$$\frac{\partial \alpha}{\partial \lambda_i} = -\langle A_i \rangle ; \quad \frac{\partial I}{\partial \langle A_k \rangle} = \lambda_k ; \quad \frac{\partial I}{\partial \lambda_i} = \sum_k^M \lambda_k \frac{\partial \langle A_k \rangle}{\partial \lambda_i}, \quad (18)$$

the last one being a generalized Fisher-Euler theorem. From (17) and (18), one can obtain an infinite set of relations linking  $I$  and  $\alpha$  by taking derivatives of (17) with respect to  $\lambda_k$  or  $\langle A_k \rangle$ . For example, the relation between the second derivatives is given by

$$\sum_{k=1}^M \left( \frac{\partial^2 I}{\partial \langle A_i \rangle \partial \langle A_k \rangle} \right) \left( \frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_j} \right) = - \sum_{k=1}^M \left( \frac{\partial \lambda_k}{\partial \langle A_i \rangle} \right) \left( \frac{\partial \langle A_j \rangle}{\partial \lambda_k} \right) = - \delta_{ij}. \quad (19)$$

where  $\delta_{ij}$  is the unit matrix. FIM expresses a relation between the independent variables or control variables (the prior information) and a dependent value  $I$ . Such information is encoded into the functional form of  $I = I(\langle A_1 \rangle, \dots, \langle A_M \rangle)$ . For later convenience, we will also denote such a relation or encoding as  $\{I, \langle A_k \rangle\}$ . We see that the Legendre transform FIM-structure involves eigenvalues of the information-Hamiltonian which display the information encoded in  $I$  via Lagrange multipliers,  $\alpha = \alpha(\lambda_1, \dots, \lambda_M)$  :

$$\{I, \langle A_k \rangle\} \quad \longleftrightarrow \quad \{\alpha, \lambda_k\}.$$

#### 4. MAIN RESULTS

We start here with our present developments. Substituting (14) into (8) and solving for  $\alpha$ , we obtain

$$\alpha = - \sum_{k=1}^M \left( 1 + \frac{k}{2} \right) \lambda_k \langle x^k \rangle. \quad (20)$$

Since  $\langle x^k \rangle$  is given by (18) as  $[-\partial \alpha / \partial \lambda_k]$ , (20) take the form

$$\alpha = \sum_{k=1}^M \left( 1 + \frac{k}{2} \right) \lambda_k \frac{\partial \alpha}{\partial \lambda_k}. \quad (21)$$

Eq. (21) constitutes an important result, since we have now at our disposal a *linear, partial differential equation (PDE)* for  $\alpha$ , whose variables are  $U(x)$ 's series-expansion's coefficients. The equation's origins are two information sources, namely, i) the Legendre structure and ii) the virial theorem. Dealing with this new equation might allow us to find  $\alpha$  in terms of the  $\lambda_k$  *without passing before through a Schrödinger equation*, a commendable achievement. See below, however, the pertinent caveats.

For convenience we now recast our key relations using dimensionless magnitudes

$$\mathcal{A} = \frac{\alpha}{[\alpha]} = \frac{\alpha}{[x]^{-2}} \quad , \quad \Lambda_k = \frac{\lambda_k}{[\lambda_k]} = \frac{\lambda_k}{[x]^{-(2+k)}} \quad , \quad (22)$$

where  $[\alpha]$  and  $[\lambda_k]$  denote the dimensions of  $\alpha$  and  $\lambda_k$ , respectively. Thus, the differential equation that governs the energy-behavior, i.e., (21), can be translated into

$$\mathcal{A} = \sum_{k=1}^M \left(1 + \frac{k}{2}\right) \Lambda_k \frac{\partial \mathcal{A}}{\partial \Lambda_k} \quad , \quad (23)$$

and is easy to obtain a solution as follows. One sets

$$\mathcal{A} = \sum_{k=1}^M \mathcal{A}_k = \sum_{k=1}^M \exp [h(\Lambda_k)] \quad , \quad (24)$$

and substitution of (24) into (23) leads to

$$\mathcal{A} = \sum_{k=1}^M \left(1 + \frac{k}{2}\right) \Lambda_k h'(\Lambda_k) \mathcal{A}_k \quad . \quad (25)$$

The above relation entails

$$h'(\Lambda_k) = \frac{2}{(2+k)} \frac{1}{\Lambda_k} \quad \longrightarrow \quad h(\Lambda_k) = \frac{2}{2+k} \ln |\Lambda_k| + d_k \quad , \quad (26)$$

where  $d_k$  is an integration constant. Finally, inserting (26) into (24) we arrive at

$$\mathcal{A} = \sum_{k=1}^M D_k \exp \left( \frac{2}{2+k} \ln |\Lambda_k| \right) \quad , \quad D_k = e^{d_k} > 0 \quad , \quad (27)$$

which can be recast as

$$\mathcal{A}(\Lambda_1, \dots, \Lambda_M) = \sum_{k=1}^M D_k |\Lambda_k|^{2/(2+k)} \quad , \quad (28)$$

or, in function of the original input-quantities (22)

$$\alpha(\lambda_1, \dots, \lambda_M) = \sum_{k=1}^M \alpha_k(\lambda_k) = \sum_{k=1}^M D_k |\lambda_k|^{2/(2+k)} \quad , \quad (29)$$

implying what seems to be a universal prescription, a linear PDE, that energy eigenvalues must necessarily comply with. This constitutes one of the main present results. Of course,

our solution poses a necessary but not (yet) sufficient condition for  $\alpha$  to be an energy-eigenvalue.

All first order, linear PDEs possess a solution that depends on an arbitrary function, called *the general solution* of the PDE. In many physical situations this solution is *less important* than other solutions called *complete ones* [21–23]. Such complete solutions are particular PDE solutions containing as many arbitrary constants as intervening independent variables. As an example we may cite the integration of the classical equations of motion via a methodology involving Hamilton-Jacobi equations, for which a complete integral is required [21–23]. We will delve into this question again in Section 8 and obtain the general solution of our PDE. In Sec. 9 we will discuss its uniqueness via analysis of the associated Cauchy problem.

## 5. MAIN PROPERTIES OF $\alpha$

Some important properties deserve special mention.

- **$\alpha$ -domain**

Obviously, it is

$$\text{Dom}[\alpha] = \{(\lambda_1, \dots, \lambda_M) / \lambda_k \in \Re\} = \Re^M$$

- **$\alpha$ -monotonicity**

Differentiating (29) we obtain

$$\frac{\partial \alpha}{\partial \lambda_k} = \frac{2}{(2+k) \lambda_k} \alpha_k = \frac{2}{(2+k) \lambda_k} D_k |\lambda_k|^{2/(2+k)} \quad (30)$$

Therefore, if  $\lambda_k < 0$ ,  $\alpha$  is a monotonically decreasing function in the  $\lambda_k$ -direction.

Also, for  $\lambda_k < 0$ , from the reciprocity relations (18) we have,

$$\langle x^k \rangle = - \frac{\partial \alpha}{\partial \lambda_k} = \frac{2}{(2+k)} D_k |\lambda_k|^{-k/(2+k)} > 0. \quad (31)$$

- **$\alpha$ -convexity**

This is a necessary property, since the  $\tilde{\alpha} = -\alpha$  is the Legendre transform of FIM.

By differentiation of the expression (30) one obtains

$$\frac{\partial^2 \alpha}{\partial \lambda_n \partial \lambda_k} = - \frac{2k}{(2+k)^2} D_k |\lambda_k|^{-2(1+k)/(2+k)} \delta_{kn}, \quad (32)$$



from which we can assert that  $\alpha$  is concave and, obviously,  $\tilde{\alpha} = -\alpha$  is a convex function. It is then guaranteed that the inverse transform of  $\partial_n \partial_k \alpha$  exists.

We end this section by mentioning that an  $I$ -analog of Eq. (29) exists, namely,

$$I(\langle x^1 \rangle, \dots, \langle x^M \rangle) = \sum_{k=1}^M I_k = \sum_{k=1}^M C_k \left| \langle x^k \rangle \right|^{-2/k}, \quad (33)$$

where  $C_k$  are positive real number (integration constant). Eq. (33) constitutes the main result of Ref. [4]. We are going to enumerate below some properties can be directly derived from it, relevant for the present work.

- **FIM-domain**

Obviously, it is

$$\text{Dom}[I] = \left\{ (\langle x^1 \rangle, \dots, \langle x^M \rangle) / \langle x^k \rangle \in \Re_o \right\}$$

- **FIM-monotonicity**

Differentiating (33) one obtain

$$\frac{\partial I}{\partial \langle x^k \rangle} = - \frac{2}{k \langle x^k \rangle} I_k = - \frac{2}{k \langle x^k \rangle} C_k \left| \langle x^k \rangle \right|^{-2/k}, \quad (34)$$

Therefore, if  $\langle x^k \rangle > 0$ ,  $I$  is a monotonically decreasing function in the  $\langle x^k \rangle$ -direction.

Also, for  $\langle x^k \rangle > 0$ , from the reciprocity relations (18) one have,

$$\lambda_k = \frac{\partial I}{\partial \langle x^k \rangle} = - \frac{2}{k} C_k \langle x^k \rangle^{-(2+k)/k} < 0. \quad (35)$$

- **FIM-convexity**

By differentiation of the expression (34) one obtains

$$\frac{\partial^2 I}{\partial \langle x^n \rangle \partial \langle x^k \rangle} = \left( \frac{2+k}{2} \right) \frac{4}{k^2} C_k \left| \langle x^k \rangle \right|^{-2(1+k)/k} \delta_{kn}, \quad (36)$$

from which we can assert that the Fisher measure is a convex function. It is then guaranteed that the inverse of  $\partial_k \partial_j \bar{\alpha}$  exists.

## 6. THE MATHEMATICAL STRUCTURE OF THE LEGENDRE TRANSFORM

In order to better understand the formalism developed in the preceding Section we scrutinize now in some detail the mathematical structure associated to the Legendre transform (see (17), (18) and (19)). This leads to a relation between the integration constants  $C_k$  and  $D_k$  pertaining to the  $I$  and  $\alpha$  expressions given by (33) and (29). We are going to study this relation in both scenarios,  $\{\alpha, \lambda_k\}$  and  $\{I, \langle x^k \rangle\}$ . Remember that our Lagrange multipliers are simultaneously  $U(x)$ 's series-expansion's coefficients.

**In a  $\{I, \langle x^k \rangle\}$  - scenario**, the  $\lambda_k$  are functions dependent on the  $\langle x^k \rangle$ -values. Taking into account (35), the energy (29) and the potential, expressed in function of the independent  $\langle x^k \rangle$ -values, take the form

$$\alpha = \sum_{k=1}^M D_k |\lambda_k|^{2/(2+k)} = \sum_{k=1}^M D_k \left( \frac{2}{k} C_k \right)^{2/(2+k)} |\langle x^k \rangle|^{-2/k}, \quad (37)$$

$$\sum_{k=1}^M \lambda_k \langle x^k \rangle = - \sum_{k=1}^M \frac{2}{k} C_k |\langle x^k \rangle|^{-2/k}. \quad (38)$$

Substituting (33), (37) and (38) into (8) we have

$$\sum_{k=1}^M C_k |\langle x^k \rangle|^{-2/k} = \sum_{k=1}^M D_k \left( \frac{2}{k} C_k \right)^{2/(2+k)} |\langle x^k \rangle|^{-2/k} - \sum_{k=1}^M \frac{2}{k} C_k |\langle x^k \rangle|^{-2/k},$$

which can be recast as

$$\sum_{k=1}^M \left\{ D_k \left( \frac{2}{k} C_k \right)^{2/(2+k)} - \frac{2+k}{k} C_k \right\} |\langle x^k \rangle|^{-2/k} = 0. \quad (39)$$

The above equation is automatically fulfilled if we impose that

$$D_k \left( \frac{2}{k} C_k \right)^{2/(2+k)} = \frac{2+k}{k} C_k,$$

which leads to

$$D_k C_k^{-k/(2+k)} = \frac{(2+k)}{2} \left( \frac{k}{2} \right)^{-k/(2+k)}. \quad (40)$$

We can verify that the above relation between  $C_k$  and  $D_k$  preserves the symmetric representation of the second derivatives (19). Using (35) we can express (32) as a function of the  $\langle x^k \rangle$ ,

$$\begin{aligned}
\frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_n} &= - \left( \frac{2}{k} \right)^{-1} \left( \frac{2}{2+k} \right)^2 D_k |\lambda_k|^{-2(1+k)/(2+k)} \delta_{kn} = \\
&= - \left( \frac{2}{k} \right)^{-(4+3k)/(2+k)} \left( \frac{2}{2+k} \right)^2 D_k C_k^{-2(1+k)/(2+k)} |\langle x^k \rangle|^{2(1+k)/k} \delta_{kn} .
\end{aligned} \tag{41}$$

The sum over  $k$  of the product of (41) and (36) leads to

$$\sum_{k=1}^M \frac{\partial^2 I}{\partial \langle x^l \rangle \partial \langle x^k \rangle} \frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_n} = - \sum_{k=1}^M \left( \frac{2}{k} \right)^{-k/(2+k)} \frac{2}{2+k} C_k^{-k/(2+k)} D_k \delta_{kn} \delta_{lk}, \tag{42}$$

which, using (40) reduces to

$$\sum_{k=1}^M \frac{\partial^2 I}{\partial \langle x^l \rangle \partial \langle x^k \rangle} \frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_n} = - \sum_{k=1}^M \delta_{kn} \delta_{lk} = - \delta_{ln}, \tag{43}$$

as expected from (19).

**In the  $\{\alpha, \lambda_k\}$  scenario**, the  $\langle x^k \rangle$  are functions that depend on the  $\lambda_k$ -values. Taking into account i) (31), ii) the FIM-relation (33), and iii) the information-potential, expressed as a function of the independent  $\lambda_k$ -values, FIM adopts the appearance

$$I = \sum_{k=1}^M C_k |\langle x^k \rangle|^{-2/k} = \sum_{k=1}^M \left( \frac{2}{2+k} \right)^{-2/k} C_k D_k^{-2/k} |\lambda_k|^{2/(2+k)}, \tag{44}$$

$$\sum_{k=1}^M \lambda_k \langle x^k \rangle = - \sum_{k=1}^M \frac{2}{(2+k)} D_k |\lambda_k|^{2/(2+k)}. \tag{45}$$

Substituting (29), (44), and (45) into (8) we have

$$\sum_{k=1}^M C_k \left( \frac{2 D_k}{2+k} \right)^{-2/k} |\lambda_k|^{2/(2+k)} = \sum_{k=1}^M D_k |\lambda_k|^{2/(2+k)} - \sum_{k=1}^M \frac{2}{(2+k)} D_k |\lambda_k|^{2/(2+k)},$$

which can be recast as

$$\sum_{k=1}^M \left\{ C_k \left( \frac{2 D_k}{2+k} \right)^{-2/k} - \frac{k}{(2+k)} D_k \right\} |\lambda_k|^{2/(2+k)} = 0.$$

The above equation is automatically fulfilled if we enforce

$$C_k \left( \frac{2 D_k}{2+k} \right)^{-2/k} = \frac{k}{(2+k)} D_k,$$

which leads to

$$C_k D_k^{-(k+2)/k} = \frac{k}{2} \left( \frac{2+k}{2} \right)^{-(k+2)/k}. \quad (46)$$

We can verify that the above relation between  $C_k$ 's and  $D_k$ 's preserves the symmetric representation of the second derivatives (19). Using (31) we can express (36) as a function of the  $\lambda_k$

$$\begin{aligned} \frac{\partial^2 I}{\partial \langle x^l \rangle \partial \langle x^k \rangle} &= \frac{(2+k)}{2} \frac{4}{k^2} C_k \left| \langle x^k \rangle \right|^{-2(1+k)/k} \delta_{lk} \\ &= \frac{(2+k)}{2} \frac{4}{k^2} C_k \left( \frac{2D_k}{2+k} \right)^{-2(1+k)/k} |\lambda_k|^{2(1+k)/(2+k)} \delta_{lk} \\ &= \left( \frac{2}{k} \right)^2 \left( \frac{2}{2+k} \right)^{-(2+3k)/k} C_k D_k^{-2(1+k)/k} |\lambda_k|^{2(1+k)/(2+k)} \delta_{lk}. \end{aligned} \quad (47)$$

The sum over  $k$  of the product of (32) and (47) now gives

$$\sum_{k=1}^M \frac{\partial^2 I}{\partial \langle x^l \rangle \partial \langle x^k \rangle} \frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_n} = - \sum_{k=1}^M \frac{2}{k} \left( \frac{2}{2+k} \right)^{-(2+k)/k} C_k D_k^{-(2+k)/k} \delta_{kn} \delta_{lk}, \quad (48)$$

which, using (46), reduces to

$$\sum_{k=1}^M \frac{\partial^2 I}{\partial \langle x^l \rangle \partial \langle x^k \rangle} \frac{\partial^2 \alpha}{\partial \lambda_k \partial \lambda_n} = - \sum_{k=1}^M \delta_{kn} \delta_{lk} = - \delta_{ln}, \quad (49)$$

as we expect from (19). From Eqs. (40) or (46) we can write

$$C_k = \frac{k}{2} \bar{C}_k, \quad D_k = \frac{k+2}{2} \bar{D}_k \quad (50)$$

with

$$\bar{D}_k^{(2+k)} = \bar{C}_k^k. \quad (51)$$

Now expressions (33) and (29) take the form,

$$I = \sum_{k=1}^M \frac{k}{2} \bar{C}_k \left| \langle x^k \rangle \right|^{-2/k}, \quad (52)$$

$$\alpha = \sum_{k=1}^M \frac{k+2}{2} \bar{D}_k |\lambda_k|^{2/(2+k)}, \quad (53)$$

and the reciprocity relations (35) and (31) are given by

$$\lambda_k = \frac{\partial I}{\partial \langle x^k \rangle} = -\bar{C}_k \langle x^k \rangle^{-(2+k)/k}, \quad (54)$$

$$\langle x^k \rangle = -\frac{\partial \alpha}{\partial \lambda_k} = \bar{D}_k |\lambda_k|^{-k/(2+k)}. \quad (55)$$

Also, we can write

$$\bar{D}_k^{(2+k)} = \bar{C}_k^k \equiv F_k^2 \quad (56)$$

then, the expressions (33) and (29), take the form,

$$I = \sum_{k=1}^M \frac{k}{2} \left[ \frac{F_k}{\langle x^k \rangle} \right]^{2/k}, \quad (57)$$

$$\alpha = \sum_{k=1}^M \frac{k+2}{2} [F_k |\lambda_k|]^{2/(2+k)}. \quad (58)$$

and the reciprocity relations (35) and (31) can be summarized as

$$F_k^2 = |\lambda_k|^k \langle x^k \rangle^{(2+k)}. \quad (59)$$

As was conjectured in [4], the reference-quantities  $F_k$  should contain important information concerning the referential system with respect the which prior conditions are experimentally determined. Following ideas advanced in [4] we will look for the “point” at which the potential function achieves a minimum.

## 7. APPROPRIATE REFERENTIAL SYSTEM

### Minimum of the information potential

It is convenient to incorporate at the outset, within the  $I$ - and  $\alpha$ -forms, information concerning the minimum of the information potential. Assume that this potential

$$U(x) = -\frac{1}{8} \sum_{x=1}^M \lambda_k x^k,$$

achieves its absolute minimum at the “critical point”  $x = \xi$

$$U'(\xi) = 0, \quad U_{min} = U(\xi). \quad (60)$$

Thus, effecting the translational transform  $u = x - \xi$  leads us to

$$I = - \sum_{k=1}^M \frac{k}{2} \lambda_k \langle x^k \rangle = - \sum_{k=1}^M \frac{k}{2} \lambda_k^* \langle u^k \rangle', \quad (61)$$

with (see the Appendix)

$$\lambda_k^* = - \frac{8}{k!} U^{(k)}(\xi), \quad \langle u^k \rangle' = \langle (x - \xi)^k \rangle \quad (62)$$

where  $U^{(k)}(\xi)$  is the  $k^{th}$  derivative of  $U(x)$  evaluated at  $x = \xi$  and  $\langle \rangle'$  indicates that the relevant moment (expectation) is computed with translation-transformed eigenfunctions.

- The corresponding FIM-explicit functional expression is built up with the  $N$ -non-vanishing momenta ( $N < M$ ) ( $\langle u^k \rangle' \neq 0$ ) and is given by

$$I = \sum_{k=2}^N \frac{k}{2} \bar{C}_k \left| \langle u^k \rangle' \right|^{-2/k} = \sum_{k=2}^N \frac{k}{2} \bar{C}_k \left| \langle (x - \xi)^k \rangle \right|^{-2/k}, \quad (63)$$

where we kept in mind that  $\lambda_1^* = -8U'(\xi) = 0$ . A glance at the above FIM-expression suggests that we re-arrange things in the fashion

$$I = \bar{C}_2 \left| \langle (x - \xi)^2 \rangle \right|^{-1} + \sum_{k=3}^N \frac{k}{2} \bar{C}_k \left| \langle (x - \xi)^k \rangle \right|^{-2/k}. \quad (64)$$

Taking now into account that

$$\begin{cases} \langle x - \xi \rangle = 0 \\ \langle (x - \xi)^2 \rangle = \langle x^2 \rangle - 2\xi \langle x \rangle + \xi^2 \end{cases} \longrightarrow \begin{cases} \langle x \rangle = \xi \\ \langle (x - \xi)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2 \end{cases} \quad (65)$$

we get

$$I = \bar{C}_2 \sigma^2 + \sum_{k=3}^N \frac{k}{2} \bar{C}_k \left| \langle (x - \xi)^k \rangle \right|^{-2/k}, \quad (66)$$

from which we obtain

$$I \sigma^2 = \bar{C}_2 + \sigma^2 \sum_{k=3}^N \frac{k}{2} \bar{C}_k \left| \langle (x - \xi)^k \rangle \right|^{-2/k} \geq 1. \quad (67)$$

Therefore, if no moment  $k \geq 3$  is a priori known, in forcing  $I$  to preserve the well-known Cramer-Rao  $I$ -bound [8]  $I \sigma^2 \geq 1$ , we need that

$$\bar{C}_2 = 1 \quad \longrightarrow \quad \bar{C}_2 = \bar{D}_2 = F_2 = 1.$$

- The corresponding  $\alpha$ -explicit functional expression is constructed with the  $N$ -non-vanishing momenta ( $N < M$ ) ( $\langle u^k \rangle' \neq 0$ ) and is given by

$$\alpha = 8 U(\xi) + \sum_{k=2}^N \frac{k+2}{2} \bar{D}_k |\lambda_k^*|^{2/(k+2)} . \quad (68)$$

For the harmonic oscillator it is well known that [4, 5]

$$U(x) = -\frac{1}{8} \lambda_2 x^2 , \quad \lambda_2 = -4\omega^2 . \quad (69)$$

The minimum of the potential function is obtained at the origin  $\xi = 0$ ,

$$V'(\xi) = -4 \lambda_2 \xi = 0 \quad \longrightarrow \quad \xi = 0 .$$

Thus, using the  $\alpha$ -expression (53) con  $\bar{D}_2 = 1$ , we have

$$\alpha = 2 |\lambda_2|^{1/2} = 4w . \quad (70)$$

as we should expect since  $(\alpha/8)$  plays the role of an energy eigenvalue [Cf. Eq. (5)] and we took Planck's constant equal to unity.

## 8. GENERAL SOLUTION OF THE ENERGY-EQUATION

Our  $\alpha$ -equation is a first order linear nonhomogeneous differential equations. We are following [21–23] in looking for the general solution. For a first-order PDE, the *method of characteristics* allows one to encounter useful curves (called characteristic curves or just characteristics) along which the PDE becomes an ordinary differential equation (ODE). Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE.

We are dealing with a first order linear nonhomogeneous equation with  $M$  independent variables of the form (22)

$$\sum_{k=1}^M \left( \frac{2+k}{2} \right) \Lambda_k \frac{\partial \mathcal{A}}{\partial \Lambda_k} = \mathcal{A} , \quad \mathcal{A} = \mathcal{A}(\Lambda_1, \dots, \Lambda_M) , \quad (71)$$

whose characteristic system

$$\frac{d\Lambda_i}{((2+i)/2)\Lambda_i} = \frac{d\Lambda_j}{((2+j)/2)\Lambda_j} = \frac{d\mathcal{A}}{\mathcal{A}} , \quad i, j = 1, \dots, M, \quad (72)$$

leads (for  $\Lambda_1 \neq 0$ ) to

$$\begin{aligned}
\frac{d\Lambda_1}{(3/2)\Lambda_1} = \frac{d\Lambda_k}{((2+k)/2)\Lambda_k} &\longrightarrow \frac{2}{3} \ln |\Lambda_1| + c_1 = \frac{2}{2+k} \ln |\Lambda_k| + c_k \\
\ln [e^{c_1} |\Lambda_1|^{2/3}] &= \ln [e^{c_k} |\Lambda_k|^{2/(2+k)}] \\
e^{c_1} |\Lambda_1|^{2/3} &= e^{c_k} |\Lambda_k|^{2/(2+k)} \\
&\downarrow \\
b_{k-1} \equiv e^{c_1 - c_k} &= \frac{|\Lambda_k|^{2/(2+k)}}{|\Lambda_1|^{2/3}} \tag{73}
\end{aligned}$$

$$\begin{aligned}
\frac{d\Lambda_1}{(3/2)\Lambda_1} = \frac{d\mathcal{A}}{\mathcal{A}} &\longrightarrow \frac{2}{3} \ln |\Lambda_1| + c_1 = \ln |\mathcal{A}| + c_{\mathcal{A}} \\
\ln [e^{c_1} |\Lambda_1|^{2/3}] &= \ln [e^{c_{\mathcal{A}}} |\mathcal{A}|] \\
e^{c_1} |\Lambda_1|^{2/3} &= e^{c_{\mathcal{A}}} |\mathcal{A}| \\
&\downarrow \\
b_M \equiv e^{c_1 - c_{\mathcal{A}}} &= \frac{|\mathcal{A}|}{|\Lambda_1|^{2/3}}. \tag{74}
\end{aligned}$$

We have now constructed an integral basis for the characteristic system (72)

$$b_1 = u_1(\Lambda_1, \dots, \Lambda_M, \mathcal{A}), \dots, b_M = u_M(\Lambda_1, \dots, \Lambda_M, \mathcal{A}), \tag{75}$$

and the general solution of equation (71) defined as

$$\Phi(u_1, u_2, \dots, u_M) = 0, \tag{76}$$

is given by

$$\Phi \left( \frac{|\Lambda_2|^{1/2}}{|\Lambda_1|^{2/3}}, \dots, \frac{|\Lambda_k|^{2/(2+k)}}{|\Lambda_1|^{2/3}}, \dots, \frac{|\Lambda_M|^{2/(2+M)}}{|\Lambda_1|^{2/3}}, \frac{|\mathcal{A}|}{|\Lambda_1|^{2/3}} \right) = 0, \tag{77}$$

where  $\Phi$  is an arbitrary function of the  $M$  variables. Solving this equation for  $\mathcal{A}$  yields a solution of the explicit form

$$\mathcal{A} = |\Lambda_1|^{2/3} \Psi \left( \frac{|\Lambda_2|^{1/2}}{|\Lambda_1|^{2/3}}, \dots, \frac{|\Lambda_k|^{2/(2+k)}}{|\Lambda_1|^{2/3}}, \dots, \frac{|\Lambda_M|^{2/(2+M)}}{|\Lambda_1|^{2/3}} \right), \tag{78}$$

where  $\Psi$  is an arbitrary function of  $(M-1)$  variables.



## 9. CAUCHY PROBLEM AND THE EXISTENCE AND UNIQUENESS OF THE SOLUTION TO OUR PDE

One of the fundamental aspects so as to have a useful PDE for modeling physical systems revolves around the existence and uniqueness of the solutions to the Cauchy problem. Here we show that such requirements are satisfied by our pertinent solutions. We start by casting (71) in the normal form

$$\frac{\partial \mathcal{A}}{\partial \Lambda_1} = F \left( \Lambda_1, \dots, \Lambda_M, \mathcal{A}, \frac{\partial \mathcal{A}}{\partial \Lambda_2}, \dots, \frac{\partial \mathcal{A}}{\partial \Lambda_M} \right) \quad (79)$$

where

$$F \left( \Lambda_1, \dots, \Lambda_M, \mathcal{A}, \frac{\partial \mathcal{A}}{\partial \Lambda_2}, \dots, \frac{\partial \mathcal{A}}{\partial \Lambda_M} \right) = \frac{2}{3\Lambda_1} \left[ \mathcal{A} - \sum_{k=2}^M \frac{k+2}{2} \Lambda_k \frac{\partial \mathcal{A}}{\partial \Lambda_k} \right], \quad (80)$$

and we see that  $F$  is a real function of class  $C^2$  in a neighborhood of

$$\Lambda_1 = a, \quad \Lambda_k = \xi_{k-1}, \quad \mathcal{A}(\xi_1, \dots, \xi_{M-1}) = c, \quad \left. \frac{\partial \mathcal{A}}{\partial \Lambda_k} \right|_{\xi_1, \dots, \xi_{M-1}} = d_{k-1}, \quad k = 2, \dots, M \quad (81)$$

Then, if  $\psi(\Lambda_2, \dots, \Lambda_M)$  is also a function of class  $C^2$  such that

$$\psi(\xi_1, \dots, \xi_{M-1}) = c, \quad \left. \frac{\partial \psi}{\partial \Lambda_k} \right|_{\xi_1, \dots, \xi_{M-1}} = d_{k-1}, \quad k = 2, \dots, M. \quad (82)$$

exists a solution  $\mathcal{A}$  of (79) in a neighborhood of  $\Lambda_1 = a$  and  $\Lambda_k = \xi_{k-1}$  that satisfies

$$\mathcal{A}(a, \Lambda_2, \dots, \Lambda_M) = \psi(\Lambda_2, \dots, \Lambda_M) \quad (83)$$

and is of class  $C^2$ .

Regarding Cauchy-uniqueness, it is known that if  $F$  satisfies the Lipschitz condition [24],

$$\begin{aligned} \left| F \left( \Lambda_1, \dots, \Lambda_M, \mathcal{A}', \frac{\partial \mathcal{A}'}{\partial \Lambda_2}, \dots, \frac{\partial \mathcal{A}'}{\partial \Lambda_M} \right) - F \left( \Lambda_1, \dots, \Lambda_M, \mathcal{A}, \frac{\partial \mathcal{A}}{\partial \Lambda_2}, \dots, \frac{\partial \mathcal{A}}{\partial \Lambda_M} \right) \right| &\leq \\ &\leq K_1 \sum_{k=2}^M \left| \frac{\partial \mathcal{A}'}{\partial \Lambda_k} - \frac{\partial \mathcal{A}}{\partial \Lambda_k} \right| + K_2 |\mathcal{A}' - \mathcal{A}| \quad K_1, K_2 = \text{const.} \end{aligned} \quad (84)$$

then, the solution of the initial value problem for (79) is unique. Note that in our case the above condition is verified always since the Legendre structure the theory guarantee that

$$F \left( \Lambda_1, \dots, \Lambda_M, \mathcal{A}, \frac{\partial \mathcal{A}}{\partial \Lambda_2}, \dots, \frac{\partial \mathcal{A}}{\partial \Lambda_M} \right) = \frac{\partial \mathcal{A}}{\partial \Lambda_1} \propto \frac{\partial \alpha}{\partial \lambda_1} = -\langle x \rangle < \infty. \quad (85)$$

## 10. CONCLUSIONS

On the basis of a variational principle based on Fisher Information we have obtained in this paper a first order differential equation for the Schrödinger energy-eigenvalues. We have shown that the general solution exists and is unique. This equation constitutes a necessary, but not sufficient condition for  $\alpha$  to be an energy-eigenvalue. Where does this equation come from?

It arises from the fact that the probability distribution that minimizes Fisher's information measure  $I$  (subject to constraints) must be derived by solving a Schrödinger-like wave equation, in which the normalization Lagrange multiplier  $\alpha$  of the associated variational problem plays the role of an energy-eigenvalue. A Legendre transform-invariant substructure emerges then that inextricably links  $I$  and  $\alpha$  as Legendre partners. This constitutes a new illustration of the power of information-related tools in analyzing physical problems.

## Appendix: FIM's translational transformation

The potential function

$$U(x) = -\frac{1}{8} \sum_{k=1}^M \lambda_k x^k.$$

can be Taylor-expanded about  $x = \xi$

$$U(x) = \sum_{k=0}^M \frac{U^{(k)}(\xi)}{k!} (x - \xi)^k.$$

The shift  $u = x - \xi$  leads to

$$\bar{U}(u) = U(u + \xi) = \sum_{k=0}^M \frac{U^{(k)}(\xi)}{k!} u^k, \quad (86)$$

which can be recast as

$$\bar{U}(u) = -\frac{1}{8} \sum_{k=0}^M \lambda_k^* u^k, \quad (87)$$

with

$$\lambda_k^* \equiv -8 \frac{U^{(k)}(\xi)}{k!} = -\frac{8}{k!} \sum_{j=1}^M j(j-1)(j-2) \cdots (j-k+1) \lambda_j \xi^{j-k}. \quad (88)$$

The shifted-FIM corresponding to  $u = x - \xi$  is obtained from (8) in the fashion [note that  $\langle \rangle'$  indicates that the pertinent moment is calculated with translation-transformed (TF) eigenfunctions]

$$I = -4 \int \psi \frac{\partial^2}{\partial x^2} \psi \, dx = -4 \int \bar{\psi} \frac{\partial^2}{\partial u^2} \bar{\psi} \, du = -4 \left\langle \frac{\partial^2}{\partial u^2} \right\rangle', \quad (89)$$

where  $\bar{\psi} = \bar{\psi}(u)$  is the TF of  $\psi(x)$ . Now, using the TF of (6) one easily finds

$$I = \int \bar{\psi}_n \left( \alpha + \sum_{k=0}^M \lambda_k^* u^k \right) \bar{\psi}_n \, du, \quad (90)$$

and one realizes that

$$I = \alpha + \sum_{k=0}^M \lambda_k^* \langle u^k \rangle' = \bar{\alpha} + \sum_{k=1}^M \lambda_k^* \langle u^k \rangle', \quad (91)$$

where

$$\bar{\alpha} = \alpha + \lambda_0^* = \alpha - 8U(\xi). \quad (92)$$

Also, the virial theorem (9) leads to

$$I = 4 \left\langle \frac{\partial^2}{\partial u^2} \right\rangle' = -4 \left\langle u \frac{\partial}{\partial u} \bar{U}(u) \right\rangle' = - \sum_{k=1}^M \frac{k}{2} \lambda_k^* \langle u^k \rangle'. \quad (93)$$

The TF moments  $\langle u^k \rangle'$  are related to the original moments as

$$\langle u^k \rangle' = \int u^k \bar{\psi}^2(u) du = \int u^k \psi^2(u + \xi) du = \int (x - \xi)^k \psi^2(x) dx = \langle (x - \xi)^k \rangle.$$

By recourse to the Newton-binomial we write

$$\int (x - \xi)^k \psi^2(x) dx = \sum_{j=1}^k (-1)^j \binom{k}{j} \xi^j \int x^{k-j} \psi^2(x) dx, \quad (94)$$

and then we finally have

$$\langle u^k \rangle' = \langle (x - \xi)^k \rangle = \sum_{j=1}^k (-1)^j \binom{k}{j} \xi^j \langle x^{k-j} \rangle. \quad (95)$$

**Acknowledgments-** This work was partially supported by the programs FQM-2445 and FQM-207 of the Junta de Andalucia-Spain, and by CONICET (Argentine Agency).

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